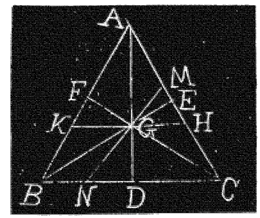
Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. R. HITT, Coronal Institute, San Marcos, Tex.

Project the given triangle into an equilateral triangle, side$=a$; let G be the center of gravity, CH or $CM=x$.

Then $BK=\dfrac{a(a-2x)}{2a-3x}$, $AK=\dfrac{a(a-x)}{2a-3x}$, $AH=a-x$.

Area $AKH=\dfrac{\triangle(a-x)^2}{a(2a-3x)}$, $BN=\dfrac{2ax-a^2}{3x-a}$,

$CN=\dfrac{ax}{3x-a}$.



Area $CMN=\dfrac{\triangle x^2}{a(3x-a)}$, area $ABNM=\triangle\left[1-\dfrac{x^2}{a(3x-a)}\right]$.

$$p=\left[\int_0^{\frac{1}{2}a}\frac{(AKH)^3dx}{\triangle^3}+\int_{\frac{1}{2}a}^{a}\frac{(ABNM)^3dx}{\triangle^3}\right]\Big/\int_0^{\frac{1}{2}a}dx$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}\left[1-\frac{x^2}{a(3x-a)}\right]^3dx+\frac{2}{a^4}\int_0^{\frac{1}{2}a}\frac{(a-x)^6dx}{(2a-3x)^3}$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}\left[1-\frac{x^2}{a(3x-a)}\right]^3dx+\frac{2}{a^4}\int_{\frac{1}{2}a}^{a}\frac{x^6dx}{(3x-a)^3}$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}dx-\frac{6}{a^2}\int_{\frac{1}{2}a}^{a}\frac{x^2dx}{3x-a}+\frac{6}{a^3}\int_{\frac{1}{2}a}^{a}\frac{x^4dx}{(3x-a)^2}$$

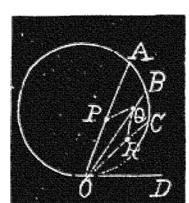
$$=\tfrac{1}{27}(\tfrac{23}{2}-\tfrac{20}{3}\log 2).$$

116. Proposed by the late ENOCH BEERY SEITZ.

The average area of the quadrilateral formed by joining four random points on the surface of a circle, radius a, is $4a^2/3\pi$.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

Let a=radius of given circle, A=its area, $\triangle$=required average, $\triangle'$=the average area when the four points are taken on both the circle A and a concentric annulus B, $\triangle_1$=the average area when three points are taken on A and one on B.



Then $(\triangle'-\triangle)A=4B(\triangle_1-\triangle)$.

But $\triangle:\triangle'=A:A+B$.

$\therefore\ \triangle'=\dfrac{(A+B)\triangle}{A}$. $\quad\therefore\ \triangle=\frac{4}{5}\triangle_1$.

Let one point O be on the circumference of the circle, and the other points P, Q, R anywhere on its surface. Let $OP=x$, $OR=y$, OQ

$=z$, $\angle AOD=\theta$, $\angle COD=\phi$, $\angle BOD=\psi$. Then $OA=2a\sin\theta=x'$, $OC=2a\sin\phi=y'$, $OB=2a\sin\psi=z'$. Area $OPQR=\frac{1}{2}xz\sin(\theta-\psi)+\frac{1}{2}yz\sin(\psi-\phi)=u$.

$$\therefore\ \triangle_1=\frac{\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'}uxyzd\theta d\phi d\psi dxdydz}{\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'}xyzd\theta d\phi d\psi dxdydz}$$

$$\therefore\ \triangle=\tfrac{4}{6}\triangle_1=\frac{24}{5\pi^3a^6}\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'}uxyzd\theta d\phi d\psi dxdydz$$

$$=\frac{32}{5\pi^3a^3}\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}[x\sin(\theta-\psi)+y\sin(\psi-\phi)]xy\sin^3\psi d\theta d\phi d\psi dxdy$$

$$=\frac{64}{15\pi^3a}\int_0^\pi\int_0^\theta\int_\psi^\theta\int_0^{x'}[3x\sin(\theta-\psi)+4a\sin\phi\sin(\psi-\phi)]\sin^2\phi\sin^3\psi d\theta d\phi d\psi xdx$$

$$=\frac{512a^2}{15\pi^3}\int_0^\pi\int_0^\theta\int_\phi^\theta[\sin\theta\sin(\theta-\psi)+\sin\phi\sin(\psi-\phi)]\sin^2\theta\sin^2\phi\sin^3\psi d\theta d\phi d\psi$$

$$=\frac{64a^2}{15\pi^2}\int_0^\pi\int_0^\theta[3(\phi-\theta)(\sin\theta\cos\theta-\sin\phi\cos\phi)+3(\sin\theta\cos\theta-\sin\phi\cos\phi)^2$$
$$+2\sin(\theta-\phi)(\sin^3\theta\cos\phi-\cos\theta\sin^3\phi)]\sin^2\theta\sin^2\phi d\theta d\phi$$

$$=\frac{2a^2}{45\pi^3}\int_0^\pi(105\theta-72\theta^2\sin\theta\cos\theta+84\theta\sin^2\theta-120\theta\sin^4\theta-105\sin\theta\cos\theta-82\sin^3\theta\cos\theta$$
$$+32\sin^5\theta\cos\theta)\sin^2\theta d\theta=4a^2/3\pi.$$

MISCELLANEOUS.

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111. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Exhibit $\cos^3\theta\sin^3\theta\sin^2\phi\cos\phi$ as a series of harmonics.

Solution by the PROPESER.

$$f(\mu,\ \phi)=\sum_{m=0}^{m=\infty}[A_{0,m}P_m(\mu)+\sum_{n=1}^{n=m}(A_{n,m}\cos n\phi+B_{n,m}\sin n\phi)P_m{}^n(\mu)].$$

Now $f(\mu,\ \phi)=\mu^3\sqrt{[(1-\mu^2)^3]}\sin^2\phi\cos\phi=\frac{1}{4}\mu^3(1-\mu^2)^{\frac{3}{2}}(\cos\phi-\cos3\phi)$.

$$A_{0,m}=\frac{2m+1}{16\pi}\int_{-1}^1\mu^3(1-\mu^2)^{\frac{3}{2}}P_m(\mu)d\mu\int_0^{2\pi}(\cos\phi-\cos3\phi)=0.$$